Math 279 Lecture 29 Notes

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1 The Final Ingredients in Our Regularity Structure

1.1 Constructing the group of transformations from a Hopf algebra

Consider a Hopf algebra $(H; \cdot, \mathbf{1}; \Delta, \mathbf{1}'; S)$ with dual $(H^*; \Delta^*, (\mathbf{1}')^*; \cdot^*, \mathbf{1}^*; S^*)$. Recall that we also have an algebra $(\mathcal{L}(H), \star, \mathbf{1} \circ \mathbf{1}')$, and recall that $S = (\mathrm{id}_H)^{-1}$ where the inverse is with respect to $\star$. Finally, we defined a map $\Gamma : H^* \to \mathcal{L}(H)$. (The example we should keep in mind is $H = T$ for the KPZ equation or for another PDE and G is a group of $\Gamma : H \to H$.) We defined $\Lambda : H^* \to \mathcal{L}(H^*)$ given by $\Lambda_g(f) = f \cdot_{\Delta^*} g$, which allowed us to define $\Gamma_g = \Lambda_g^*$.

Observe that $\Lambda_{g_1 \cdot_{\Delta^*} g_2} = \Lambda_{g_2} \circ \Lambda_{g_1}$. From this, we can readily deduce that $\Gamma_{g_1 \cdot_{\Delta^*} g_2} = \Gamma_{g_1} \circ \Gamma_{g_2}$. In other words, we have $\Gamma : (H^*, \cdot_{\Delta^*}) \to (\mathcal{L}(H), \circ)$ as a homomorphism with respect to these algebra structures. For our purposes, we need a group. Namely, define the group of characters

$$G_0 = \{g \in H^* : g : H \to \mathbb{R} \text{ linear}, g(h_1 \cdot h_2) = g(h_1)g(h_2), g(\mathbf{1}) = 1\}.$$

We can see[1] that if $g_1, g_2 \in G_0$, then $g_1 \cdot_{\Delta^*} g_2 \in G_0$. It turns out that if $g \in G_0$ and $\widehat{g} = g \circ S \in G_0$ then $\widehat{g} \cdot_{\Delta^*} g = g \cdot_{\Delta^*} \widehat{g} = (\mathbf{1}')^*$.

Recall that a **graded** bialgebra has $\mathcal{H} = \bigoplus_n \mathcal{H}_n$ with $\cdot : \mathcal{H}_m \otimes \mathcal{H}_m \to \mathcal{H}_{n+m}$ and $\Delta : \mathcal{H}_n \to \bigoplus_{i+j=n} \mathcal{H}_i \otimes \mathcal{H}_j$, and recall that a **connected** bialgebra has $\mathcal{H}_0 = \{\lambda \mathbf{1} : \lambda \in k\}$.

Theorem 1.1. *Any connected, graded bialgebra has a unique antipode S.*

Proof. Here is the idea: Let $u_0 = \mathbf{1} \circ \mathbf{1}'$ denote the unit for $(\mathcal{L}(H), \star)$. We want to say something like

$$\begin{aligned}(\mathrm{id})^{-1} &= (u_0 - (u_0 - \mathrm{id}))^{-1} \\ &= \sum_{k \geq 0} (u_0 - \mathrm{id})^{\star k}.\end{aligned}$$

[1] Professor Rezakhanlou uses the phrase "It is not hard to see." He has gotten comments from referees on his papers saying that he should prove more things. I like to avoid that phrase because sometimes it is hard to see.

This is algebra; we can't have an infinite sum! All we need to verify is that the if $h \in \mathcal{H}_n$, then $(\sum_{k\geq 0}(u_0 - \mathrm{id})^{\star k})(h) = \sum_{k=0}^{n}(u_0 - \mathrm{id})^{\star k}$ for some n. This is where the graded condition comes in. $\square$

Example 1.1. Let $H = T(\mathbb{R}^\ell) = \bigoplus_{n\geq 0} H_n$, with $H_n = \operatorname{span}\{e_{i_1,\cdots,i_n} : i_1,\dots,i_n \in \{1,\dots,\ell\}\}$. The product on H is the **shuffle product**, $e_I \sqcup\!\sqcup e_J$. The coproduct is $\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_i (v_0 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n)$ with $v_0 = 1$. We are interested in $\mathbf{x} : [0,T]^2 \to H^*$ with

$$\langle \mathbf{x}, e_{i_1,\dots,i_n}\rangle = \int_0^T \cdots \int_0^T dx^{i_1}\cdots dx^{i_n}.$$

As we discussed before, we require two properties:

1. $\mathbf{x}$ is a character:

$$\mathbf{x}(a \sqcup\!\sqcup b) = \langle \mathbf{x}, a \sqcup\!\sqcup b\rangle = \mathbf{x}(a)\mathbf{x}(b) = \langle \mathbf{x}, a\rangle\langle \mathbf{x}, b\rangle.$$

2. Chen's relation:

$$\mathbf{x}(s,u) \cdot_{\Delta^*} \mathbf{x}(u,t) = \mathbf{x}(s,t), \quad \text{so} \quad \mathbf{x}(s,t) = (\mathbf{x}(s))^{-1} \cdot_{\Delta^*} \mathbf{x}(t).$$

Now we want to use the same idea for the KPZ equation, but it doesn't work exactly the same way. We need a small variation of what we have done so far so we can deal with functions and distributions separately. Namely, we have two spaces $(T, \cdot)$ (algebra) and $(T^*, \cdot, \widetilde{\Delta}^+)$, where $\widetilde{\Delta}^+$ is a suitable coproduct. Moreover, we need $\Delta^+ : T \to T \otimes T^+$.

Recall that $\Gamma_g = (\mathrm{id} \otimes g) \circ \Delta$. Now given $g \in (T^+)^*$, we define $\Gamma_g(h) = (\mathrm{id} \otimes g) \circ \Delta^+(h)$ (so $\Gamma_g : T \to T$). Again, we may define

$$G^+ = \{g \in (T^+)^* : g(h_1 \cdot h_2) = g(h_1)g(h_2), g(\mathbf{1}) = 1\},$$

which is a group. As before, we have

$$\Gamma_{g_1 \cdot_{\Delta^+} g_2} = \Gamma_{g_1} \circ \Gamma_{g_2}.$$

Given the pair (T, T^+) with Δ^+, we are ready to build our regularity structure (not just for KPZ but for all the examples that have been worked out in this context). We use the following scheme:

First, build a linear $\mathbf{\Pi} : T \to \mathcal{D}'$, and imagine we have a map $F : \mathbb{R}^d \to G = \{\Gamma_g : G \in G^+\}$, so $F(x) = \Gamma_{f(x)}$. Then we set $\Pi_x = \mathbf{\Pi} \circ F_x^{-1}$. Then the requirement that $\Pi_x \Gamma_{x,y} = \Pi_y$ or $\mathbf{\Pi} F_x^{-1}\Gamma_{x,y} = \mathbf{\Pi} F_y^{-1}$ leads to $\Gamma_{x,y} = F_y^{-1} \circ F_x$. In fact, our T is the algebra generated by $\mathbf{1}, X_1, X_2, \partial_\ell \mathscr{I}(\tau), \mathscr{I}(\tau), \dots$. T^+ is the algebra freely generated by $\mathbf{1}, X_1, X_2, ((\partial_\ell \mathscr{I})(\tau) : 2 - \deg \tau - |\ell| > 0)$. Set

$$\Delta^+(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \qquad \Delta^+(X_i) = \mathbf{1} \otimes X_i + X_i \otimes \mathbf{1}, \qquad \Delta^+(\Xi) = \Xi \otimes \mathbf{1},$$

$$\Delta^+(\underbrace{\tau\cdot\tau'}_{\text{product in } T}) = \underbrace{\Delta^+(\tau)\cdot\Delta^+(\tau')}_{\text{product in } T^+},$$

$$\Delta^+\mathscr{I}(\tau) = (\mathscr{I}\otimes\mathrm{id})\Delta^+\tau + \sum_{\ell:|\ell|<\deg\tau+2} \frac{X^\ell}{\ell!}\otimes\partial_\ell\mathscr{I}(\tau).$$

1.2 Renormalization and the Wick product

We carry our all these operations to build a suitable operator $H:\mathbb{R}_x\times[0,T)\to R$ which depends on the model we have for ξ. Now replace ξ with the smoothized version $|xi*\chi^\varepsilon$ and denote the correspoding solution by H^ε. However, H^ε does not converge as $\varepsilon\to 0$. For example, replace Ξ by ξ^ε and consider $\partial\mathscr{I}(\xi^\varepsilon)$; this does not converge as $\varepsilon\to 0$. The issue is $(K_x*\xi^\varepsilon)^2\to\infty$, where K is the heat kernel.

However, a miracle happens. If we look at $(K_x-\xi^\varepsilon)^2-\mathbb{E}[(K_x*\xi^\varepsilon)^2]$, this has a limit as $\varepsilon\to 0$. We have

$$\int f(z_1,z_2)\xi(z_1)\xi(z_2)\,dz_1\,dz_2,$$

which causes a problem, and we replace it by

$$\int f(z_1,z_2)\xi(z_1)\diamond\xi(z_2)\,dz_1\,dz_2,$$

where $\diamond$ is the **Wick product**:

$$\xi(z_1)\diamond\xi(z_2)=\xi(z_1)\xi(z_2)-\delta_0(z_1-z_2).$$

It turns out that all that we need to do is subtract a constant. These constants lie in a 4 or 6 dimensional group, but in the original problem, we only see 1 dimension of this.